Local Stabilization of Analytic Systems with n-1 inputs

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Abstract In this communication, we investigate the local stabilization of analytic systems $\dot{x} = f(x) + Bu$ with $n-1$ inputs. We state a sufficient condition on the first approximation of f for which the local asymptotic stabilizability can be achieved; furthermore, we give explicitly the stabilizing feedback.

Keywords Feedback, asymptotic stabilization, nonlinear systems, Lyapunov's function.

1 Introduction

We consider a nonlinear control system with $n-1$ inputs:

$$\left\{ \begin{array}{l} \dot{x} = f(x) + Bu \\ x \in \mathbb{R}^n, \quad u \in \mathbb{R}^{n-1} \end{array} \right. \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is an analytic mapping, B is a $n \times (n-1)$ matrix of rank $n-1$.

We shall say that system (1) is locally asymptotically stabilizable (L.A.S.) if there exists (locally) a feedback $u : \mathbb{R}^n \to \mathbb{R}^{n-1}$ such that $u(0) = 0$ and the origin is a locally asymptotically state equilibrium point for the closed-loop system $\dot{x} = f(x) + Bu(x)$.

Local stabilizability has been investagated by many authors: one can cite Kawski([7]), Dayawansa-Martin-Knowles ([3],[4]), Boothby-Marino ([1]) who studied planar systems as well as Hermes ([5]) .

In a precedent work ([6]), we gave necessary and sufficient conditions for a homogeneous polynomial system to be globally stabilizable. In this paper, we give a sufficient condition for (1) to be L.A.S. and we construct explicitly the stabilizing feedback. As an example, consider the system:

$$\left\{ \begin{array}{rcl} \dot{x}_1 &=& e^{x_1+x_3} - e^{x_2+x_3} - x_1 - x_2 \\ \dot{x}_2 &=& u_2 \\ \dot{x}_3 &=& u_3 \end{array} \right.$$

We shall see that it can be stabilized by means of the following feedback law:

$$\left\{ \begin{array}{rcl} u_2 &=& -(1+2x_1)(e^{x_1} - e^{x_2} - x_1 + x_2) \\ && -\, x_1 \left(1 - e^{x_2} \dfrac{\exp(x_1 + x_1^2 + x_2) - 1}{x_1 + x_1^2 + x_2} \right) \\ && -\, x_1 - x_1^2 - x_2 \\ u_3 &=& -\dfrac{e^{x_3} - 1}{x_3} (e^{x_1} - e^{x_2}) \cdot \\ && \big(x_1 + (x_1 + x_1^2 + x_2)(2x_1 + 1) \big) \end{array} \right.$$

The linearized system of (1) is:

$$\dot{x} = Ax + Bu \tag{2}$$

where $A = \dfrac{\partial f}{\partial x}(0)$. It is well known that a sufficient condition for (1) to be L.A.S. is that (2) is also L.A.S. ([2]) but what happens when one has $A \equiv 0$? We can generally write $f(x) = P_r(x) + o(|\ x\ |^r)$ where P_r, the first approximation of f, is a homogeneous polynomial of degree r. In the sequel, we shall give conditions on polynomial P_r for which system (1) is L.A.S.; in fact we shall see that if the system:

$$\dot{x} = P_r(x) + Bu \tag{3}$$

is stabilizable then so is system (1). The particular case when f is exactly a homogeneous polynomial was studied in [6].

2 Notations and preliminaries

We denote by $\langle x, y \rangle$ the usual inner product and by det the determinant. For $n-1$ vectors $x_1, \ldots, x_{n-1}$, we denote by $x_1 \wedge \ldots \wedge x_{n-1}$ the unique vector satisfying:

$$\forall x \in \mathbb{R}^n \quad \det(x_1, \ldots, x_{n-1}, x) = \langle x_1 \wedge \ldots \wedge x_{n-1}, x\rangle$$

We denote by $b_2, \ldots, b_n$ the columms of matrix B and by b_1, the vector $b_1 = b_2 \wedge \ldots \wedge b_n$; clearly $(b_1, b_2, \ldots, b_n)$ is a basis of $\mathbb{R}^n$ in which system (1) can be written:

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & f_1(x) \\ \dot{x}_2 & = & f_2(x) + u_2 \\ \vdots & & \\ \dot{x}_n & = & f_n(x) + u_n \end{array}\right. \tag{4}$$

after the following change in inputs space:

$$u_i = -f_i(x) + v_i \quad i = 2, \ldots, n$$

system (4) becomes:

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & f_1(x) \\ \dot{x}_2 & = & v_2 \\ \vdots & & \\ \dot{x}_n & = & v_n \end{array}\right. \tag{5}$$

f_1 is an analytic function, so it can be written:

$$f_1(x) = \sum_{i=r}^{\infty} P_i(x)$$

when the P_i are homogeneous polynomials of degree i.

3 Case where r is even

Theorem 1 *If the polynomial P_r takes both positive and negative values, then system (5) is stabilizable.*

Proof

Since $P_r(x)$ changes its sign, there exists $\tilde{b}_1$ and e_2 such that $P_r(\tilde{b}_1) \cdot P_r(e_2)$ is negative. Let $H = \mathrm{span}(b_2, \ldots, b_n)$, H has an empty interior so we can suppose that $\tilde{b}_1$ does not belong to H. The plane spaned by vectors $\tilde{b}_1$, and e_2 intersects H at $\tilde{b}_2$ and without loss of generality we can assume that $(\tilde{b}_2, b_3, \ldots, b_n)$ is a basis of H. The transformation matrix between the basis $(b_1, b_2, \ldots, b_n)$ and $(\tilde{b}_1, \tilde{b}_2, b_3, \ldots, b_n)$ is:

$$T = \begin{pmatrix} \alpha_1 & 0 & 0 & \ldots & 0 \\ \alpha_2 & \beta_2 & 0 & \ldots & 0 \\ \alpha_3 & \beta_3 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ \alpha_n & \beta_n & 0 & \ldots & 1 \end{pmatrix}$$

$\alpha_1, \ldots, \alpha_n, \beta_2, \ldots, \beta_n$ being such that $\tilde{b}_1 = \sum_{i=1}^n \alpha_i b_i$ and $\tilde{b}_2 = \sum_{i=2}^n \beta_i b_i$

We have:

$$T^{-1} = \begin{pmatrix} 1/\alpha_1 & 0 & 0 & \ldots & 0 \\ -\alpha_2/\beta_2\alpha_1 & 1/\beta_2 & 0 & \ldots & 0 \\ \star & & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ \star & \star & 0 & \ldots & 1 \end{pmatrix}$$

In the new basis $(\tilde{b}_1, \tilde{b}_2, b_3, \ldots, b_n)$ system (5) is given by the set of equations:

$$\left\{\begin{array}{lcl} \dot{\tilde{x}}_1 & = & \tilde{f}_1(\tilde{x}) \\ \dot{\tilde{x}}_2 & = & \tilde{f}_2(\tilde{x}) + v_2/\beta_2 \\ \dot{\tilde{x}}_3 & = & \tilde{f}_3(\tilde{x}) + \gamma_3 v_2 + v_3 \\ \vdots & & \\ \dot{\tilde{x}}_n & = & \tilde{f}_n(\tilde{x}) + \gamma_n v_2 + v_n \end{array}\right. \tag{6}$$

where $\tilde{f}_1(\tilde{x}) = \dfrac{1}{\alpha_1} f_1(T\tilde{x})$, $\tilde{f}_1$ can be written:

$$\tilde{f}_1(\tilde{x}) = g(\tilde{x}_1, \tilde{x}_2) + \sum_{i=3}^{n} \tilde{x}_i g_i(\tilde{x})$$

where $g_i(\tilde{x}) = \displaystyle\int_0^1 \frac{\partial \tilde{f}}{\partial \tilde{x}_i}(\tilde{x}_1, \tilde{x}_2, t\tilde{x}_3, \ldots, t\tilde{x}_n)\, dt$

and $g(\tilde{x}_1, \tilde{x}_2) = \tilde{f}_1(\tilde{x}_1, \tilde{x}_2, 0, \ldots 0) = \displaystyle\sum_{i=r}^{\infty} \tilde{P}_i(\tilde{x}_1, \tilde{x}_2)$ the $\tilde{P}_i$ being homogeneous polynomials of degree i, and $\tilde{P}_r$ taking both positive and negative values.

We focus our attention on the system:

$$\left\{\begin{array}{lcl} \dot{\tilde{x}}_1 & = & \tilde{P}_r(\tilde{x}_1, \tilde{x}_2) + \tilde{P}_{r+1}(\tilde{x}_1, \tilde{x}_2) + \cdots \\ \dot{\tilde{x}}_2 & = & w_2 \end{array}\right. \tag{7}$$

$\tilde{P}_r(\tilde{x}_1, \tilde{x}_2)$ can be written:

$$\tilde{P}_r(\tilde{x}_1, \tilde{x}_2) = L_1^{r_1} L_2^{r_2} \cdots L_q^{r_q} Q_1^{m_1} \cdots Q_s^{m_s}$$

where $L_1, L_2, \ldots, L_q$ are linear forms two by two linearly independant and $Q_1, \ldots, Q_s$ are irreductible homogeneous polynomials of degree 2. Since $\tilde{P}_r$ changes its sign, at least two exponents r_i and r_j are odd; we can assume that $i = 1$, $j = 2$ and since L_1 and L_2 are linearly independant, we can suppose that $L_1 = \lambda\tilde{x}_1 + \mu\tilde{x}_2$ with $\mu \neq 0$. Let $h(\tilde{x}_1, \tilde{x}_2, z) = \displaystyle\int_0^1 \frac{\partial g}{\partial \tilde{x}_2}(\tilde{x}_1, \tilde{x}_2 - tz)\, dt$ and consider the feedback law:

$$\begin{array}{lcl} w_2(\tilde{x}_1, \tilde{x}_2) & = & \dfrac{-\lambda + \gamma(1 + 1/r_1)\tilde{x}_1^{1/r_1}}{\mu} g(\tilde{x}_1, \tilde{x}_2) \\ & & - \tilde{x}_1 h(\tilde{x}_1, \tilde{x}_2, z) - z \end{array}$$

together with the positive definite function:

$$V(\tilde{x}_1, \tilde{x}_2) = \frac{1}{2}(\tilde{x}_1^2 + z^2)$$

where we put

$$\begin{array}{lcl} z & = & \tilde{x}_2 - k(\tilde{x}_1) \\ k(\tilde{x}_1) & = & \dfrac{-\lambda\tilde{x}_1 + \gamma\tilde{x}_1^{1+1/r_1}}{\mu} \end{array}$$

Taking into account that $g(\tilde{x}_1,\tilde{x}_2) = g(\tilde{x}_1, k(\tilde{x}_1) + zh(\tilde{x}_1,\tilde{x}_2,z)$, an easy computation shows that $\dot{V}$, the derivative of V along the trajectories of closed-loop system (7), is equal to:

$$\dot{V} = \tilde{x}_1 g(\tilde{x}_1, k(\tilde{x}_1)) - z^2$$

In order to show that 0 is an asymptotically equilibrium point for closed-loop system (7),it suffices to prove that one can choose γ such that:

$$\tilde{x}_1 g(\tilde{x}_1, k(\tilde{x}_1)) < 0 \tag{8}$$

Since for $i \neq 1$, L_i is linearly independant of L_1 and the Q_i s are irreductible, one has $\tilde{P}_r(\tilde{x}_1, k(\tilde{x}_1)) = \gamma^{r_1} A\tilde{x}_1^{r+1} + \mathrm{o}(\mid \tilde{x}_1^{r+1} \mid)$ with A a nonzero constant.

For $s \geq r+2$, $\tilde{P}_s$ is a homogeneous polynomial of degree s so $\tilde{P}_s(\tilde{x}_1, k(\tilde{x}_1)) = \mathrm{o}(\mid \tilde{x}_1^{s-1} \mid) = \mathrm{o}(\mid \tilde{x}_1^{r+1} \mid)$, $\tilde{P}_{r+1}$ can be written:

$$\tilde{P}_{r+1}(\tilde{x}_1,\tilde{x}_2) = \sum_{i=0}^{r+1} \alpha_i \tilde{x}_1^i \tilde{x}_2^{r+1-i}$$

so

$$\tilde{P}_{r+1}(\tilde{x}_1, k(\tilde{x}_1)) = \sum_{i=0}^{r+1} \alpha_i \tilde{x}_1^{r+1} \left(\frac{-\lambda + \gamma \tilde{x}_1^{1/r1}}{\mu} \right)^{r+1-i}$$

therefore, one can write:

$$g(\tilde{x}_1,\tilde{x}_2) = (\gamma^{r_1} A + B)\tilde{x}_1^{r+1} + \mathrm{o}(\tilde{x}_1^{r+1})$$

hence if γ is chosen such that $\gamma^{r_1} A + B < 0$, inequality (8) holds in a neighborhood of the origin because $r+1$ is odd.

Let us return now to system (6) and consider the following feedback law:

$$\begin{cases} v_2(\tilde{x}) &= -\beta_2 \tilde{f}_2(\tilde{x}) + w_2(\tilde{x}_1,\tilde{x}_2) \\ v_3(\tilde{x}) &= \tilde{f}_3(\tilde{x}) - \gamma_3 v_2(\tilde{x}) - \dfrac{\partial V}{\partial \tilde{x}_1} g_3(\tilde{x}) - \tilde{x}_3 \\ \vdots & \\ v_n(\tilde{x}) &= \tilde{f}_n(\tilde{x}) - \gamma_n v_2(\tilde{x}) - \dfrac{\partial V}{\partial \tilde{x}_1} g_n(\tilde{x}) - \tilde{x}_n \end{cases}$$

together with the positive definite function:

$$W(\tilde{x}) = V(\tilde{x}_1,\tilde{x}_2) + \frac{1}{2}(\tilde{x}_3^2 + \cdots + \tilde{x}_n^2)$$

Clearly the derivative of W along the trajectories of closed-loop system (6) is:

$$\dot{W} = \tilde{x}_1 g(\tilde{x}_1, k(\tilde{x}_1)) - z^2 - \tilde{x}_3^2 - \cdots - \tilde{x}_n^2$$

which is obviously negative definite in a neighborhood of the origin. This proves that the above feedback stabilizes system (6).

4 Case where r is odd

Theorem 2 *If there exists $(\lambda_2,\ldots,\lambda_n) \in \mathbb{R}^{n-1}$ such that $P_r(1,\lambda_2,\ldots,\lambda_n) < 0$ then system (5) is stabilizable.*

Proof Consider the following system derived from system (5):

$$\begin{cases} \dot{x}_1 &= P_r(x) \\ \dot{x}_2 &= u_2 \\ \vdots & \\ \dot{x}_n &= u_n \end{cases} \tag{9}$$

We claim that the following feedback law stabilizes system (8):

$$u_i(x) = \lambda_i P_r(x) - x_1 g_i(x) - (x_i - \lambda_i x_1)^r$$

where $g_i(x) = \displaystyle\int_0^1 \frac{\partial P_r}{\partial x_i}(x_1, t(x_2,\ldots,x_n) + x_1(1-t)(\lambda_2,\ldots,\lambda_n))\,dt$
Indeed if we introduce the positive definite function:

$$V(x) = \frac{1}{2}\left(x_1^2 + \sum_{i=2}^{n} (x_i - \lambda_i x_1)^2 \right)$$

taking into account that $P_r(x) = P_r(x_1, \lambda_2 x_1, \ldots, \lambda_n x_1) + \sum_{i=2}^n (x_i - \lambda_i x_1) g_i(x)$, we obtain that the derivative of V along the trajectories of closed-loop system (8) is:

$$\dot{V} = x_1^{r+1} P_r(1,\lambda_2,\ldots,\lambda_n) - \sum_{i=2}^{n} (x_i - \lambda_i x_1)^{r+1}$$

which is negative definite. Now, one can remark that the u_i are homogeneous polynomials of the same degree as P_r, so a Massera's theorem ([8]) permits to conclude about L.A.S. of closed-loop system (5) with the feedback given above.

References

[1] W. Boothby and R. Marino. Feedback stabilization of planar nonlinear systems. *Systems & Control Letters*, (12):87–92, (1989).

[2] R.W. Brockett. *Differential Geometric Control Theory*, chapter Asymptotic stability and feedback stabilization, pages 181–191. Brockett, Milmann, Sussmann, 1983.

[3] W.P. Dayawansa and C.F. Martin. Asymptotic stabilization of two-dimensional real analytic systems. *Systems & Control Letters*, (12):205–211, (1989).

[4] W.P. Dayawansa, C.F. Martin, and G. Knowles. Asymptotic stabilization of a class of smooth two-dimensional systems. *SIAM J. Control and Optimization*, (28):1321–1349, (1990).

[5] H. Hermes. Homogeneous coordinates and continuous stabilizing feedback controls. In *Differential Equations. Stability and Control.* S. Elaydi, 1990.

[6] A. Iggidr and J.C. Vivalda. Global stabilization of homogeneous polynomial systems in $\mathbb{R}^n$. *Nonlinear Anal. TMA.* to appear.

[7] M. Kawski. Stabilization of nonlinear systems in the plane. *Systems & Control Letters*, (12):169–175, (1990).

[8] J.L. Massera. Contribution to stability theory. *Annals of Mathematics*, (64):182–206, (1956).